A TYPE OF THERMOHYDRODYNAMIC GAS LENS

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The focusing properties of a laminar gas stream in a circular cylindrical tube with a linearly varying wall temperature are studied and optical calculations are carried out in approximation of geometric optics.

Particular attention has recently been devoted to problems of developing long-range communications systems involving the use of lasers [1–6]. However, the distribution of the refractive index in the study of gas lenses and lightguides is generally not coordinated with that actually attainable. In this paper we study a possible variant of a gas lens intended for the transmission of a light beam with small losses and simultaneous focusing. We will study those gas lenses whose focusing action is based on the relationship between the refractive index n of the gas and its temperature. At constant pressure we have

$$n-1=n^*=n_0^*\frac{T_0}{T}. \tag{1}$$

Here $n_0 = n_0^* + 1$ is the refractive index of the gas at the temperature T_0. The propagation of the light beam in a medium with a variable refractive index will be examined within the framework of geometric optics. The trajectory of the beam is defined [9] by

$$\frac{1}{\rho}=\frac{1}{n}\,\mathbf{N}\ \mathrm{grad}\ n. \tag{2}$$

The beam is bent in the direction of increasing refraction. Thus the design of gas lenses is associated with the development of appropriate temperature fields within the closed channels.

Let us examine the flow of a gas in a straight circular tube of radius R and of length L. Here it is assumed that: 1) the flow is laminar; 2) the velocity profile is after Poiseuille; 3) energy dissipation is neglected; 4) the thermodynamic characteristics of the gas are independent of temperature.

Having introduced the dimensionless coordinates x and y, the velocity v and the temperature θ, we write the energy equation [7]

$$(1-y^2)\frac{\partial\theta}{\partial x}=\beta\left[\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial\theta}{\partial y}\right)+\frac{\partial^2\theta}{\partial x^2}\right], \tag{3}$$

where

$$\beta=a/\nu\,\mathrm{Re}. \tag{4}$$

It is easy to prove that the following will be one of the solutions [8]:

$$\theta=bx-c(1-y^2)+\frac{c}{4}(1-y^4), \tag{5}$$

where b and c are constants expressed in terms of β:

$$b=4\beta c. \tag{6}$$

The coefficient c is defined by the radial temperature difference ΔT in any cross section of the tube

$$c=\frac{4}{3}\,\frac{\Delta T}{T_0}. \tag{7}$$

As is seen from (6) and (7), for a linearly varying wall temperature the radial temperature profile is

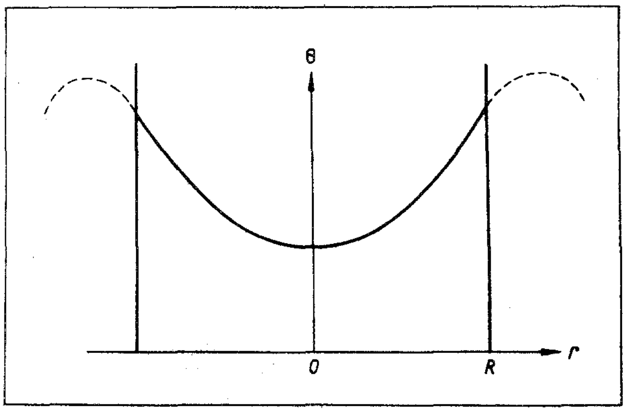

Fig. 1. Radial temperature distribution in tube.

defined by the longitudinal temperature gradient of the wall. The gas temperature increases monotonically from the axis to the wall. Figure 1 shows the form of the temperature profile at any of the lateral cross sections of the tube (all profiles are similar).

Let us return to the optical portion of the problem. We will reverse the direction of the x-axis, since the beam must be directed in a direction opposite to that of the longitudinal temperature gradient. Formula (5) is rewritten to the form

$$\theta=-bx-c(1-y^2)+\frac{c}{4}(1-y^4). \tag{8}$$

In view of axial symmetry we can limit ourselves to examination of the plane problem. Equation (2), with consideration of (1) and (8), assumes the form

$$y''+\frac{n_0^*c}{(1+\theta)(1+\theta+n_0^*)}\times$$

$$\times(1+y'^2)(4\beta y'+2y-y^3)=0. \tag{9}$$

The solution of (9) for the boundary conditions

$$y(x=0)=y_0,\quad y'(x=0)=y_0' \tag{10}$$

yields the trajectory y = y(x).

Let us evaluate the order of magnitude for the coefficients in (9). For gases $n^*\sim 10^{-3}$–10^{-4}. When

$T_0 \sim 300$–$320°$ K and $\Delta T \sim 5$–$25°$ K the coefficient c falls within the range 0.02–0.1, $\beta \sim 1/\mathrm{Re}$, since a and ν are of the same order, $-1 < \theta < 0$.

Below we will limit ourselves throughout to paraxial beams for which

$$|y_0| \ll 1, \quad |y_0'| \ll 1. \tag{11}$$

With y'^2 and n_0^* negligibly small in comparison with unity, instead of (10) we derive the equation

$$(1+\theta)^2 y'' + c^*(4\beta y' + 2y - y^3) = 0, \tag{12}$$

where $c^* = n_0^* c$.

Let us examine some approximations simplifying the solution of (12) and making it possible to obtain an analytic solution.

I. Let us assume $|\theta| \ll 1$ and neglect the term c^*y^3 in (12). This indicates that we are speaking of small relative temperature variations. Then

$$y'' + 4\beta c^* y' + 2c^* y = 0. \tag{13}$$

For the condition

$$\lambda^2 = 2c^*(1 - 2\beta^2 c^*) > 0 \tag{14}$$

which, obviously, will be satisfied, the trajectory described by (12) has the form

$$y = A\exp(-2\beta c^* x)\sin\lambda(x + B). \tag{15}$$

Here

$$A = y_0\sqrt{1 + \frac{(2\beta c^* y_0 + y_0')^2}{\lambda^2 y_0^2}}, \tag{16}$$

$$\lambda B = \mathrm{arc\,tg}\,\frac{\lambda y_0}{2\beta c^* y_0 + y_0'}. \tag{17}$$

The beam trajectory as is seen from (15) (in the adopted approximation) oscillates about the lens axis. The oscillations are damped, but insignificantly, since the exponent is small. From (15) we can find the relationship between y_0 and y_0' at which the beam fails to reach the lens wall. Analysis shows that considerably more rigorous limitations are imposed on the divergence of the entry beam than on its initial deviation from the lens axis.

The oscillation period is given by

$$\tilde{x} = \frac{2\pi}{\lambda}, \tag{18}$$

and it diminishes as the radial temperature gradient increases.

II. If in (12) we again assume $|\theta| \ll 1$ and neglect the term proportional to y', since it results in no significant changes in trajectory, we obtain the equation

$$y'' + c^*(2y - y^3) = 0. \tag{19}$$

The solution of this equation is expressed in terms of the elliptical Jacobi functions [10]. The beam trajectory is a periodic curve. The oscillation period is

$$\tilde{x} = 4\sqrt{2/(2+A)c^*}\,K(k), \tag{20}$$

where

$$K(k) = \int_0^{\pi/2} (1 - k^2\sin^2\varphi)^{-1/2} d\varphi, \tag{21}$$

and the modulus of the elliptical function

$$k^2 = \frac{2 - A}{2 + A}, \tag{22}$$

$$A^2 = 4 - \left(4y_0^2 - y_0^4 + \frac{2}{c^*}\,y_0'^2\right). \tag{23}$$

The permissible values of k^2 lie in the interval

$$0 \leqslant k^2 \leqslant 1/3, \tag{24}$$

which imposes limitations on y_0 and y_0'.

III. If in (12) we neglect the terms proportional to y' and y^3, but allow for the change in temperature along the axis, assuming $\theta \approx -bx$, we derive the equation

$$(1 - bx)^2 y'' + 2c^* y = 0. \tag{25}$$

If the condition

$$g^2 = \frac{n_0^*}{8\beta^2 c} - \frac{1}{4} > 0 \tag{26}$$

is satisfied, Eq. (25) has a solution of the following form [11]:

$$y = A\sqrt{1 - 4\beta cx}\,\sin\ln[g(1 - 4\beta cx) + B], \tag{27}$$

with A and B integration constants.

Condition (26) is satisfied, for example, for air when $\mathrm{Re}^2/c > 10^4$, i.e., even when $\mathrm{Re} > 10^2$.

Analysis of (27) and comparison of the latter with (15) demonstrates that the oscillation periods in both

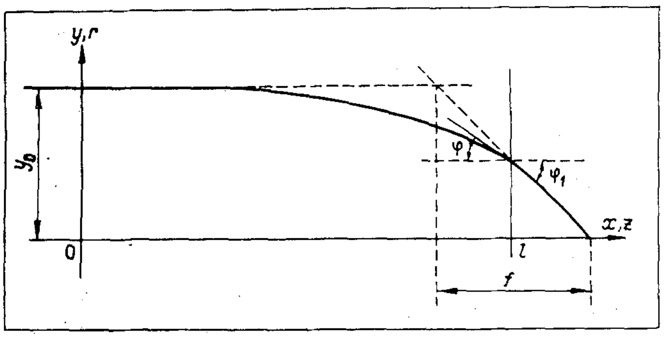

Fig. 2. Determination of gas lens focal length.

cases are close to each other, but consideration of the longitudinal temperature gradient yields considerably more pronounced attenuation of the beam-trajectory oscillation amplitude than the solution of (15).

Knowing the trajectory of the beams in the gas lens, we can calculate its focal length f. Let n_1 be the refractive index of the medium surrounding the lens. The beam enters the lens parallel to its axis at a distance y_0 (see Fig. 2). Passing through the lens and refracted at the boundary between the lens and the external medium, the beam is propagated rectilinearly. The difference between the abscissas of the points of beam intersection with the x-axis, and the intersection of the extension of the latter with the initial direction, is taken as the focal length of the lens.

From Fig. 2 we find

$$f = \left| \frac{y_0}{\operatorname{tg} \varphi_1} \right|, \quad n[l,\ y(l)] \sin\varphi = n_1 \sin\varphi_1.$$

Assuming the angles φ and φ_1 to be small, we have

$$\operatorname{tg}\varphi = y'(l) \approx \sin\varphi.$$

Then

$$f = \frac{n_1}{n[l,\ y(l)]} \left| \frac{y_0}{y'(l)} \right|.$$

It is obvious that the lens will focus those beams for which $y'(l) \gtrless 0$ when $y(l) \gtrless 0$. Since the refractive index of the gas varies only slightly, even with considerable changes in temperature, with extremely great accuracy we find that

$$\frac{n_1}{n[l,\ y(l)]} \approx 1,$$

so that

$$f = \left| \frac{y_0}{y'(l)} \right|.$$

Since the beam trajectory is a function of y_0, the lens will exhibit spherical aberration.

NOTATION

Here x is the dimensionless coordinate along the lens axis; y is the dimensionless coordinate along the radius; T_0 is the tube wall temperature at $x = 0$; θ is the relative temperature change; u is the gas velocity; u_0 is the gas velocity at the tube axis; l is the dimensionless lens length; f is the focal length based on tube radius; a is the thermal diffusivity; ν is the kinematic viscosity; c_p is the heat capacity at constant pressure; ρ is the radius of light beam curvature.

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